Weak Integer Additive Set-Indexers of Certain Graph Operations

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Abstract

An integer additive set-indexer is defined as an injective function $f : V(G) \to 2^{\mathbb{N}_0}$ such that the induced function $g_f : E(G) \to 2^{\mathbb{N}_0}$ defined by $g_f(uv) = f(u)+f(v)$ is also injective, where $f(u)+f(v)$ is the sum set of $f(u)$ and $f(v)$ and $\mathbb{N}_0$ is the set of all non-negative integers. If $g_f(uv) = k\ \forall\ uv \in E(G)$, then f is said to be a k-uniform integer additive set-indexers. An integer additive set-indexer f is said to be a weak integer additive set-indexer if $|g_f(uv)| = max(|f(u)|, |f(v)|)\ \forall\ uv \in E(G)$. We have some characteristics of the graphs which admit weak integer additive set-indexers. In this paper, we study the admissibility of weak integer additive set-indexer by certain finite graph operations.

Key words: Integer additive set-indexers, weak integer additive set-indexers, mono-indexed elements of a graph, sparing number of a graph.
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1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [6], [1], and [2]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

An *integer additive set-indexer* (IASI, in short) is defined in [3] as an injective function $f : V(G) \to 2^{\mathbb{N}_0}$ such that the induced function $g_f : E(G) \to 2^{\mathbb{N}_0}$ defined by $g_f(uv) = f(u) + f(v)$, where $f(u) + f(v)$ is the sumeset of $f(u)$ and $f(v)$, is also injective. If $g_f(e) = k\ \forall\ e \in E(G)$, then f is called a k-uniform IASI.

The cardinality of the labeling set of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element.

The characteristics of weak IASI graphs have been done in [4] and [5]. The following are the major notions and results established in these papers.

Lemma 1.1. *[4] For an integer additive set-indexer f of a graph G, we have $max(|f(u)|, |f(v)|) \leq |g_f(uv)| = |f(u) + f(v)| \leq |f(u)||f(v)|$, where $u, v \in V(G)$.*

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Definition 1.2. [4] An IASI f is said to be a *weak IASI* if $|g_f(uv)| = max(|f(u)|, |f(v)|)$ for all $u, v \in V(G)$. A graph which admits a weak IASI may be called a *weak IASI graph*.

Definition 1.3. [5] An element (a vertex or an edge) of graph which has the set-indexing number 1 is called a *mono-indexed element* of that graph. The *sparing number* of a graph G is defined to be the minimum number of mono-indexed edges required for G to admit a weak IASI and is denoted by $\varphi(G)$.

Theorem 1.4. *[7] If a graph G is a weak IASI graph, then any subgraph H of G is also a weak IASI graph. Or equivalently, if G is a graph which has no weak IASI, then any supergraph of G does not have a weak IASI.*

Theorem 1.5. *[7] If a connected graph G admits a weak IASI, then G is bipartite or G has at least one mono-indexed edge. Hence, all paths, trees and even cycles admit a weak IASI. We observe that the sparing number of bipartite graphs is* 0.

Theorem 1.6. *[7] The complete graph K_n admits a weak IASI if and only if the number of edges of K_n that have set-indexing number* 1 *is* $\frac{1}{2}(n-1)(n-2)$.

Theorem 1.7. *[7] An odd cycle C_n has a weak IASI if and only if it has at least one mono-indexed edge.*

Theorem 1.8. *[7] Let C_n be a cycle of length n which admits a weak IASI, for a positive integer n. Then, C_n has an odd number of mono-indexed edges when it is an odd cycle and has even number of mono-indexed edges, when it is an even cycle.*

2 Weak IASI of Graph Operations

In this section, we discuss the admissibility of weak IASI to certain operations of graphs.

In fact, the intersection of paths or cycles or both is a path and hence by Remark 1.5, it admits a weak IASI. For finite number of cycles $C_{n_1}, C_{n_2}, C_{n_3}, \ldots, C_{n_r}$ which admit weak IASIs, their intersection $\bigcap_{i=1}^{r} C_{n_i}$ admits a weak IASI if all cycles C_{n_i} have a common path.

Given two graphs G_1 and G_2, the intersection $G_1 \cap G_2$ need not be a path. If G_1 and G_2 admit weak IASIs, say f_1 and f_2 respectively, then their intersection $G_1 \cap G_2$ admits a weak IASI if and only if f_1 and f_2 are suitably defined in such a way that $f = f_1|_{G_1 \cap G_2} = f_2|_{G_1 \cap G_2}$, where $f_i|_{G_1 \cap G_2}, i = 1, 2$, is the restriction of f_i to $G_1 \cap G_2$.

2.1 Weak IASI of the Union of Graphs

Definition 2.1. [1] The union $G_1 \cup G_2$ of two graphs (or two subgraphs of a given graph) $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is the graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$. If G_1 and G_2 are disjoint graphs, then their union is called *disjoint union* of G_1 and G_2.

The union of two graphs we mention here need not be the disjoint union. First, we discuss the admissibility of weak IASI by the union of two graphs G_1 and G_2.

Theorem 2.2. *Let G_1 and G_2 be two cycles. Then, $G_1 \cup G_2$ admits a weak IASI if and only if both G_1 and G_2 are weak IASI graphs.*

Proof. Let G_1 and G_2 be two weak IASI graphs. Let $f_1 : V(G_1) \to 2^{\mathbb{N}_0}$ be a weak IASI for G_1 and $f_2 : V(G_2) \to 2^{\mathbb{N}_0}$ be a weak IASI for G_2.

If G_1 and G_2 be two disjoint cycles, then define $f : V(G_1 \cup G_2) \to 2^{\mathbb{N}_0}$ by

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in G_1 \\ f_2(v) & \text{if } v \in G_2 \end{cases}$$

.

If G_1 and G_2 be two graphs with some common elements, then define f as above, with an additional condition that $f_1 = f_2 = f$ for all the elements in $G_1 \cap G_2$. Therefore, f is a weak IASI for $G_1 \cup G_2$.

Conversely, assume that $G_1 \cup G_2$ is a weak IASI graph. Then, both G_1 and G_2 are subgraphs of $G_1 \cup G_2$. Hence, by Theorem 1.4, both G_1 and G_2 admit weak IASIs. □

In the following theorem we discuss about the sparing number of the union of two weak IASI graphs.

Theorem 2.3. *Let G_1 and G_2 be two weak IASI graphs. Then, $\varphi(G_1 \cup G_2) = \varphi(G_1) + \varphi(G_2) - \varphi(G_1 \cap G_2)$.*

Proof. Let G_1 and G_2 be two weak IASI graphs with the corresponding weak IASIs f_1 and f_2 respectively. Define a function $f : G_1 \cup G_2 \to 2^{\mathbb{N}_0}$, such that

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in G_1 \\ f_2(v) & \text{if } v \in G_2 \end{cases}$$

Then,

$$\begin{aligned} G_1 \cup G_2 &= (G_1 - (G_1 \cap G_2)) \cup (G_2 - (G_1 \cap G_2)) \cup (G_1 \cap G_2) \\ \varphi(G_1 \cup G_2) &= \varphi(G_1) - \varphi(G_1 \cap G_2) + \varphi(G_2) - \varphi(G_1 \cap G_2) + \varphi(G_1 \cap G_2) \\ &= \varphi(G_1) + \varphi(G_2) - \varphi(G_1 \cap G_2). \end{aligned}$$

This completes the proof. □

2.2 Weak IASI of the Join of Graphs

Definition 2.4. [6] Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then, their *join* (or *sum*), denoted by $G_1 + G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup E_{ij}$, where $E_{ij} = \{u_i v_j : u_i \in G_1, v_j \in G_2\}$.

In this section, we verify the admissibility of a weak IASI by the join of paths, cycles and graphs. We proceed by using the following notion.

The graph $P_n + K_1$ is called a *fan graph* and is denoted by $F_{1,n}$. The following result establishes the admissibility of weak IASI by a fan graph $F_{1,n}$.

Theorem 2.5. *Let $F_{1,n} = P_n + K_1$. Then, $F_{1,n}$ admits a weak IASI if and only if P_n is 1-uniform or K_1 is mono-indexed.*

Proof. Assume that P_n is 1-uniform. Denote the single vertex in K_1 by v. If we label v by a singleton set, then $F_{1,n}$ is 1-uniform. If label v by a non-singleton vertex, then every edge in $F_{1,n}$ has at least one mono-indexed vertex. This labeling is a weak IASI for $F_{1,n}$. Assume that P_n is not 1-uniform. If K_1 is mono-indexed, then the corresponding set-label in $F_{1,n}$ is a weak IASI.

Conversely, assume that $F_{1,n}$ is a weak IASI graph. If P_n is 1-uniform, then the proof is complete. Hence, assume that P_n is not 1-uniform. Then, at least one vertex of P_n must have a non-singleton set-label. Therefore, K_1 must be mono-indexed, since $F_{1,n}$ admits a weak IASI.

This completes the proof. □

From Theorem 2.5, we note that the number of mono-indexed edges in $F_{1,n}$ is minimum when K_1 is mono-indexed. Hence, we have the following result.

Proposition 2.6. *The sparing number of a fan graph $F_{1,n} = P_n + K_1$ is $\lceil \frac{n-1}{2} \rceil$.*

The following theorem establishes the admissibility of the join of two paths in a given graph G.

Theorem 2.7. *Let P_m, P_n be two paths. Then, the join $P_m + P_n$ admits a weak IASI if and only if P_m or P_n is a 1-uniform graph.*

Proof. By Remark 1.5, all paths admit weak IASI. Without loss of generality, assume that P_m is 1-uniform. Let $E_{ij} = \{uv : u \in P_m, v \in P_n\}$. Then, every edge in E_{ij} has at least one mono-indexed vertex. Hence, $P_m + P_n$ admits a weak IASI. Conversely, assume that the join $P_m + P_n$ of two paths P_m and P_n admits a weak IASI. If P_m and P_n are not 1-uniform, then neither of the end vertices of some edge e in E_{ij} are mono-indexed, which is a contradiction to the hypothesis. Hence, either P_m or P_n must be 1-uniform. □

Definition 2.8. A *wheel graph* W_n is a graph with n vertices, $(n \geq 4)$, formed by connecting all vertices of an $(n-1)$-cycle C_{n-1} to a single vertex other than the vertices of C_{n-1}. That is, $W_n = K_1 + C_{n-1}$.

Theorem 2.9. *Let C_n be a cycle of length n which has a weak IASI. Then, the wheel graph $W_{n+1} = C_n + K_1$ admits a weak IASI if and only if C_n is 1-uniform or K_1 is mono-indexed.*

Proof. Let v be the single vertex of K_1. If we label v by a non-singleton set, then, as W_{n+1} has a weak IASI, no vertex of C_n can have a non-singleton set-label. That is, C_n is 1-uniform. Conversely, if C_n is 1-uniform, then W_{n+1} is a weak IASI graph for any set-label of K_1.

Next, assume that C_n is not 1-uniform. Let W_{n+1} is a weak IASI graph. Since v is adjacent to every vertex of C_n, it can only have a singleton set-label. That is, K_1 is mono-indexed. Conversely, If we label v by a singleton set, then, since C_n has a weak IASI, it forms a weak IASI for W_{n+1}.

Hence, the wheel graph $W_{n+1} = C_n + K_1$ admits a weak IASI if and only if C_n is 1-uniform or K_1 is mono-indexed. □

From Theorem 2.9, we note that the number of mono-indexed edges in W_{n+1} is minimum when K_1 is mono-indexed. Hence, we have the following proposition.

Proposition 2.10. *The sparing number of a wheel graph $W_{n+1} = C_n + K_1$ is $\lceil \frac{n-1}{2} \rceil$.*

Theorem 2.11. *Let C_n be a cycle that admits a weak IASI and P_m be a path. Then, their join $G = C_n + P_m$ admits a weak IASI if and only if either C_n or P_m is a 1-uniform IASI graph.*

Proof. First, assume that either C_n or P_m is a 1-uniform IASI graph. Then, every edge u_iv_j in $G = C_n + P_m$, where $u_i \in P$ and $v_j \in C_n$ has at least one mono-indexed end vertex. Then, such a labeling is a weak IASI for G.

Conversely, assume that $G = C_n + P_m$ admits a weak IASI. If possible, assume that neither C_n nor P_m is 1-uniform. Let u_i be a vertex in P_m and v_j be a vertex in C_n which have set-indexing numbers greater than 1. Since every vertex of P_m is adjacent to every vertex of C_n in G, we have an edge u_iv_j in G whose both the end vertices have set-indexing number greater than 1, which is a contradiction to the hypothesis. Therefore, either P_m or C_n must be 1-uniform. □

Theorem 2.12. *Let C_m and C_n be two cycles which admit weak IASIs. Then $C_m + C_n$ admits a weak IASI if and only if all elements of either C_m or C_n are mono-indexed. In other words,the join $C_m + C_n$ of two weak IASI cycles C_m and C_n, admits a weak IASI if and only if either C_m or C_n is a 1-uniform IASI graph.*

Proof. Without loss of generality, let all elements of the cycle C_m are mono-indexed. Also, let the cycle C_n admits a weak IASI. Then, every edge in $C_m + C_n$ has at least one mono-indexed end vertex. Therefore, $C_m + C_n$ admits a weak IASI.

Conversely, Assume that $C_m + C_n$ admits a weak IASI. If possible, assume that there exist some elements (vertices or edges) in both C_m and C_n which are not mono-indexed. Let u_i be a vertex in C_m and v_j be a vertex in C_n which are not mono-indexed. Then, the edge u_iv_j in $C_m + C_n$ has both the end vertices having set-indexing number greater than 1, which is a contradiction to the hypothesis. Hence, either C_m or C_n must have all its elements mono-indexed. □

The following result is a more general result of the above theorems.

Theorem 2.13. *Let G_1 and G_2 be two weak IASI graphs. Then, the graph $G_1 + G_2$ is a weak IASI graph if and only if either G_1 or G_2 is a 1-uniform IASI graph.*

In fact, we can generalise Theorem 2.12, to the join of finite number of cycles as given in the following theorem.

Theorem 2.14. *Let $C_{n_1}, C_{n_2}, C_{n_3}, \ldots, C_{n_r}$ be r cycles which admit weak IASIs. Then, their join $\sum_{i=1}^{r} C_{n_i}$ admits a weak IASI if and only if all cycles C_{n_i}, except one, are 1-uniform IASI graphs.*

Proof. Let $G = \sum_{i=1}^{r} C_{n_i}$. Without loss of generality, assume that all cycles, except C_1, are 1-uniform. Then, all edges in the graph G have at least one mono-indexed end vertex . That is, G is a weak IASI graph.

Conversely, C is a weak IASI Graph. Since every vertex of each cycle is adjacent to the vertices of all other cycles, the vertices of C that are not mono-indexed must belong to the same cycle. Therefore, all cycles in C, except one, are 1-uniform. □

Furthermore, we observe that Theorem 2.14 is true not only for finite cycles in a given graph G, but for finite number of graphs too. Hence, we propose the following result.

Theorem 2.15. *Let $G_1, G_2, G_3, \ldots\ldots, G_n$ be weak IASI graphs. Then, the graph $\sum_{i=1}^{n} G_i$ is a weak IASI graph if and only of all given graphs G_i, except one, are 1-uniform IASI graphs.*

Admissibility of weak IASI by graph joins have been discussed so far. Now, we proceed to discuss about the sparing number of these graphs. The following results provide the sparing number of the join of two paths or cycles which admit weak IASI.

Proposition 2.16. *Let P_m and P_n be two paths, where $m < n$. Then, the sparing number of G is given by*

$$\varphi(P_m + P_n) = \begin{cases} \frac{m}{2}(n+2) & \text{if } P_n \text{ is even.} \\ \frac{m}{2}(n+1) & \text{if } P_n \text{ is odd.} \end{cases}$$

Proof. Let P_m and P_n be two paths of lengths m and n respectively. Let $m < n$. By Theorem 2.15, $P_m + P_n$ is a weak IASI graph if and only if either P_m or P_n is 1-uniform. Since $m < n$, let P_m be 1-uniform.

Let P_n be of even length. Then, P_n has $\frac{n}{2}$ mono-indexed edges connecting P_m and P_n. Therefore, there are $m.\frac{n}{2}$ mono-indexed edges. Hence, the total number of mono-indexed edges is $m + m.\frac{n}{2} = \frac{m}{2}(n+2)$.

Let Let P_n be of odd length. Therefore, P_n has $\frac{(n-1)}{2}$ mono-indexed edges connecting P_m and P_n. Hence, the total number of mono-indexed edges is $m + m.\frac{n-1}{2} = \frac{m}{2}(n+1)$. Therefore, there are $\frac{m}{2}(n+1)$ mono-indexed edges. □

Proposition 2.17. *Let C_m and C_n be two cycles, where $m < n$. Then, the sparing number of $C_m + C_n$ is given by*

$$\varphi(C_m + C_n) = \begin{cases} \frac{m}{2}(n+2) & \text{if } C_n \text{ is even.} \\ 1 + \frac{m}{2}(n+3) & \text{if } C_n \text{ is odd.} \end{cases}$$

Proof. Let C_m and C_n be two cycles, where $m < n$. By Theorem 2.15, $C_m + C_n$ is a weak IASI graph if and only if either C_m or C_n is 1-uniform. Since $m < n$, let C_m be 1-uniform.

Let C_n be an even cycle. Then, C_n has $\frac{n}{2}$ mono-indexed edges connecting C_m and C_n. But, C_n need not have any mono-indexed edge. Therefore, there are $m.\frac{n}{2}$ mono-indexed edges. Hence, the total number of mono-indexed edges in C_m+C_n is $m+m.\frac{n}{2} = \frac{m}{2}(n+2)$.

Let C_n be of odd length. Then C_n has (at least) one mono-indexed edge and has $\frac{(n+1)}{2}$ mono-indexed edges connecting C_m and C_n. Hence, the total number of mono-indexed edges is $1 + m + m.\frac{n+1}{2} = 1 + \frac{m}{2}(n+3)$. Therefore, there are $1 + \frac{m}{2}(n+1)$ mono-indexed edges in $C_m + C_n$. □

In a similar way, we can establish the following result also.

Proposition 2.18. *Let P_m be a path and C_n be a cycle. If $m < n$, then the sparing number of $P_m + C_n$ is given by*

$$\varphi(P_m + C_n) = \begin{cases} \frac{m}{2}(n+2) & \text{if } C_n \text{ is even.} \\ 1 + \frac{m}{2}(n+3) & \text{if } C_n \text{ is odd.} \end{cases}$$

If $m > n$, then the sparing number of $P_m + C_n$ is given by

$$\varphi(P_m + C_n) = \begin{cases} \frac{n}{2}(m+2) & \text{if } P_m \text{ is of even length.} \\ \frac{n}{2}(m+1) & \text{if } P_m \text{ is of odd length.} \end{cases}$$

2.3 Weak IASI of the Ring sum of Graphs

Definition 2.19. [2] Let G_1 and G_2 be two graphs. Then the *ring sum* (or *symmetric difference*) of these graphs, denoted by $G_1 \oplus G_2$, is defined as the graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \oplus E_2$, leaving all isolated vertices, where $E_1 \oplus E_2 = (E_1 \cup E_2) - (E_1 \cap E_2)$.

Remark 2.20. Let P_m and P_n be two paths in a given graph G. Then, $P_m \oplus P_n$ is a path or disjoint union of paths or a cycle. Hence, $P_m \oplus P_n$ admits a weak IASI if it is a path or disjoint union of paths or an even cycle and admits a weak IASI with at least one mono-indexed edge if it is an odd cycle.

Remark 2.21. Let P_m be a path and C_n be a cycle in a given graph G. If P_m and C_n are edge disjoint, then $P_m \oplus C_n = P_m \cup C_n$. Therefore, $P_m \oplus C_n$ admits a weak IASI if and only if C_n has a weak IASI. If P_m and C_n have some edges in common, then $P_m \oplus C_n$ is a path. Hence, by Theorem 1.5, $P_m \oplus C_n$ admits a weak IASI.

The following theorem establishes the admissibility of weak IASI by the ring sum of two cycles.

Theorem 2.22. *If C_m and C_n are two cycles which admit weak IASIs, and $C_m \oplus C_n$ be the ring sum of C_m and C_n. Then,*

(i) if C_m and C_n are of same parity, $C_m \oplus C_n$ admits a weak IASI.

(ii) if C_m and C_n are of different parities, $C_m \oplus C_n$ admits a weak IASI if and only if it has odd number of mono-indexed edges.

Proof. Let C_m and C_n be two cycles which admit weak IASIs. If C_m and C_n have no common edges, then $C_m \oplus C_n = C_m \cup C_n$. This case has already been discussed in the previous section. Hence, assume that C_m and C_n have some common edges.

Let v_i and v_j be the end vertices of the path common to C_m and C_n. Let $P_r, r < m$ be the (v_i, v_j)-section of C_m and $P_s, s < n$ be the (v_i, v_j)-section of C_n, which have no common elements other than v_i and v_j. Hence, we have $C_m \oplus C_n = P_r \cup P_s$ is a cycle. Then, we have the following cases.

Case 1: Let C_m and C_n are odd cycles. If C_1 and C_2 have an odd number of common edges, then both P_r and P_s are paths of even length. Hence, the cycle $P_r \cup P_s$ is an even cycle. Therefore, $C_m \oplus C_n$ has a weak IASI. If C_m and C_n have an even number of common edges, then both P_r and P_s are paths of odd length. Therefore, the cycle $P_r \cup P_s$ is an even cycle. Hence, $C_m \oplus C_n$ has a weak IASI.

Case 2: Let C_m and C_n are even cycles. If C_m and C_n have an odd number of common edges, then both P_r and P_s are paths of odd length. Hence, the cycle $P_r \cup P_s$ is an even cycle. Therefore, $C_m \oplus C_n$ has a weak IASI. If C_m and C_n have an even number of common edges, then both P_r and P_s are paths of even length. Hence, the cycle $P_r \cup P_s$ is an even cycle. Therefore, $C_m \oplus C_n$ has a weak IASI.

Case 3: Let C_m and C_n be two cycles of different parities. Without loss of generality, assume that C_m is an odd cycle and C_n is an even cycle. Let C_m and C_n have an odd number of common edges. Then, the path P_r has even length and the path P_s has odd length. Hence, the cycle $P_r \cup P_s$ is an odd cycle. Therefore, by Theorem 1.7, $C_m \oplus C_n$ has a weak IASI if and only if $P_r \cup P_s$ has odd number of edges of set-indexing number 1. Let C_m and C_n have an even number of common edges. Then, P_r has odd length and P_s has even length. Hence, the cycle $P_r \cup P_s$ is an odd cycle. therefore, by Theorem 1.7, $C_m \oplus C_n$ has a weak IASI if and only if $P_r \cup P_s$ has odd number of edges of set-indexing number 1. □

Definition 2.23. Let H be a subgraph of the given graph G, then $G \oplus H = G - H$, which is called *complement of H in G*.

Therefore, we have the following proposition on the complement of a subgraph of G in G.

Theorem 2.24. *Let G be a weak IASI graph. Then, the complement of any subgraph H in G is also a weak IASI graph under the induced weak IASI of G.*

Proof. Let G admits a weak IASI,say f. Let H be a subgraph of the graph G. The complement of H in G, $G - H = G \oplus H$, is a subgraph of G. Hence, as G is a weak IASI graph, by Theorem 1.4, the restriction of f to $G - H$ is a weak IASI for $G - H$. □

2.4 Weak IASI of the Complements of Graphs

In this section, we report some results on the admissibility of weak IASI by the complements of different weak IASI graphs and their sparing numbers. We also discuss about the sparing number of self-complementary graphs.

A graph G and its comlement $\bar{G}$ have the same set of vertices and hence G and $\bar{G}$ have the same set-labels for their corresponding vertices. The set-labels of the vertices in $V(G)$ under a weak IASI of G need not form a weak IASI for the complement of G. A set-labeling of $V(G)$ that defines a weak IASI for both the graphs G and its complement $\bar{G}$ may be called a *concurrent set-labeling*. The set-labels of the vertex set of G mentioned in this section are concurrent.

Proposition 2.25. *Let G be a bipartite graph and let $\bar{G}$ be its complement. Then, $\bar{G}$ is a weak IASI graph if and only if G and $\bar{G}$ are a 1- uniform IASI graphs.*

Proof. Let G be a bipartite graph. Then, it is a weak IASI graph with bipartition of the vertex set (X_1, X_2). If G is 1-uniform, then every vertex of G is mono-indexed. Hence, its complement $\bar{G}$ is also 1-uniform. Therefore, $\bar{G}$ is also a weak IASI graph.

Conversely, assume that $\bar{G}$ is a weak IASI graph. Now, let X_1 be the set of all mono-indexed vertices in G and if possible, let X_2 be the set of all vertices of G having set-indexing number greater than 1. Then, $\bar{G}$ consists of two cliques, one is the graph G_1 induced by X_1 and other is the graph G_2 induced by X_2. clearly, the G_1 is 1-uniform. If G_2 is not 1-uniform, then each vertex of G_2 have set-indexing number greater than 1, which is a contradiction to the hypothesis that $\bar{G}$ has a weak IASI. Then, both G_1 and G_2 are 1-uniform components of $\bar{G}$. That is, each vertex in $V(G)$ is mono-indexed. Hence, G and $\bar{G}$ are 1-uniform IASI graphs. □

Now, we proceed to verify the admissibility of weak IASI by the complements of cycles. As a result, we have the following theorem.

Proposition 2.26. *Let C_n be a cycle on n vertices. Then, its complement $\bar{C_n}$ admits a weak IASI if and only if C_n has at most one vertex of set-indexing number greater than 1.*

Proof. We have $C_n \cup \bar{C_n} = K_n$. If $\bar{C_n}$ is a weak IASI graph, then by Theorem 2.2, K_n is also a weak IASI graph. Then by Theorem 1.6, at most one vertex of C_n can have a set-indexing number greater than 1. Conversely, let at most one vertex of C_n (and $\bar{C_n}$) has a vertex of set-indexing number greater than 1. Then, every edge of C_n and $\bar{C_n}$ has at least one end vertex that is mono-indexed. Hence, $\bar{C_n}$ is a weak IASI graph. □

Corollary 2.27. *Let C_n be a cycle on n vertices. If C_n and its complement $\bar{C_n}$ are weak IASI graphs, then the minimum number of mono-indexed edges in $\bar{C_n}$ is $\frac{1}{2}n(n-3)$.*

Proof. If C_n and its complement $\bar{C_n}$ are weak IASI graphs, then by Proposition 2.26, C can have at most one vertex of set-indexing number greater than 1. That is, C can have at most 2 edges that is not mono-indexed. Hence, by Theorem 1.6, $\bar{C}$ contains at least $\frac{1}{2}(n-1)(n-2) - 2 = \frac{1}{2}n(n-3)$ edges. □

Corollary 2.28. *Let G be an r-regular weak IASI graph. If its complement $\bar{G}$ is also a weak IASI graph, then $\bar{G}$ contains at least $\frac{1}{2}[(n-1)(n-2) - 2r]$ mono-indexed edges.*

Proof. Let G be an r-regular graph. Then, its complement $\bar{G}$ admits a weak IASI if and only if G can have at most one vertex that is not mono-indexed. Therefore, since G is r-regular, it can have at most r edges that are not mono-indexed. Hence, $\bar{G}$ contains at least $\frac{1}{2}(n-1)(n-2) - r = \frac{1}{2}[(n-1)(n-2) - 2r]$ mono-indexed edges. □

Proposition 2.29. *Let G be a connected weak IASI graph on n vertices. If its complement $\bar{G}$ is also a weak IASI graph, then $\bar{G}$ contains at least $\frac{1}{2}[(n-1)(n-2) - 2r]$ mono-indexed edges, where $r = \Delta(G)$, the maximum vertex degree.*

Proof. Let G be an r-regular graph. Let v be a vertex in G of degree $r = \Delta(G)$. The complement $\bar{G}$ of G admits a weak IASI if and only if G can have at most one vertex that is not mono-indexed. If we label v by an r-element set, G has r edges that are not mono-indexed. That is, G can have at most r mono-indexed edges. Hence, $\bar{G}$ contains at least $\frac{1}{2}(n-1)(n-2) - r = \frac{1}{2}[(n-1)(n-2) - 2r]$ mono-indexed edges. □

An interesting question that arises here is about the number of mono-indexed edges in a self-complementary, weak IASI graph. The following results address this problem.

Proposition 2.30. *If G is a self-complementary r-regular graph on n vertices which admits a weak IASI, then G and $\bar{G}$ contain at least $\frac{1}{2}r(2r-1)$ mono-indexed edges.*

Proof. Since, the vertices of G and $\bar{G}$ have the same set-labels and $G \cup \bar{G} = K_n$, by Theorem 1.6, at most one vertex of G and $\bar{G}$ can have a non-singleton set-label. Label a vertex of G, say v, by a non-singleton set. Then, since G is r-regular, r edges incident on v are not mono-indexed. That is, G has at most r edges that are not mono-indexed. Since G is self-complementary, $E(G) = E(\bar{G})$ and $E(G) \cup E(\bar{G}) = E(K_n) = \frac{n(n-1)}{2}$. $|E(G)| = \frac{n(n-1)}{4}$. Since G and $\bar{G}$ have at most r edges that are not mono-indexed, the minimum number of mono-indexed edges in G is $\frac{n(n-1)}{4} - r = \frac{1}{4}[n(n-1) - 4r]$. But, since $G \cong \bar{G}$ and $G \cup \bar{G} = K_n$, degree of v in $G \cup \bar{G}$ is $(n-1)$. Hence, $2r = n-1$. Therefore, the minimum number of mono-indexed edges in G is $\frac{1}{4}[n(n-1) - 4r] = \frac{1}{4}[(2r+1)2r - 4r] = \frac{1}{2}r(2r-1)$. □

Remark 2.31. We note that C_5 is the only cycle that is self-complementary. That is, C_5 is the only graph that is 2-regular and self-complementary. Hence, C_5 or its complement can have at least 3 mono-indexed edges under the IASI which is a weak IASI for both of them.

If G is not r-regular, the number of mono-indexed edges in G and $\bar{G}$ need not be equal. The relation between number of mono-indexed edges in G and $\bar{G}$ is given in the following proposition.

Proposition 2.32. *If G is a self-complementary graph on n vertices which has l mono-indexed edges, then the number of mono-indexed edges in $\bar{G}$ is $n - l - 1$.*

3 Conclusion

In this paper, we have discussed the admissibility of certain finite graph operations. More properties and characteristics of weak IASIs, both uniform and non-uniform, are yet to be investigated. We have formulated some conditions for some graph classes and graph operations to admit weak and strong IASIs. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain IASIs still remain unsettled. All these facts highlight a wide scope for further studies in this area.

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